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Quaternion Form of Longitude Latitude and Heading Kinematics Equations

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Abstract: The kinematics equations of longitude and latitude have singularities in polar of the earth. To solve the problem, a solution called the quaternion form of longitude, latitude and heading kinematics equations was created and introduced in the paper. The key point of the solution is to define an instantaneous great circle for a moving particle. To a moving particle, it is impossible to define three definite Euler angles, thus the definite quaternion to it does not exist. But to the instantaneous great circle, three definite Euler angles can be defined. Meanwhile, the instantaneous great circle is rotating by driving of the moving particle, thus quaternion can be used to model the instantaneous great circle. The model is the kinematics equations of longitude, latitude and heading in quaternion form. This form of equations can be used all over of the earth. It works well on the polar of the earth automatically. Verifying by mathematics simulation has been designed and practiced. The simulation includes some flights around the earth with flying by the polar and turning in polar region. The results of simulation suggest that the flight plan can be executed precisely by the algorithm. The solution can be applied in fields of flight simulation and inertial navigation.

Keywords: Navigation, Flight Simulation, Polar, Quaternion, Longitude, Latitude, Heading

1. Introduction

When studying a moving ship or a flying aircraft, it is important to model their position in space.

The combination of longitude, latitude and altitude is the most important set to describe the position of a static or moving particle in the technology fields of navigation and aerospace.

To an inertial navigation device which is a physical system, acquiring the position in space includes two parts of work. The first part is measuring accelerations and angular rates. The second part is computing attitude, velocity and position by mathematics models. The computing models can be classified into two levels, the dynamics level and the kinematics level. The whole process of measuring and computing is called mechanization in navigation technique field [5].

To a simulating system, such as a flight simulator, the work of measurement is not necessary. Thus, the two fields may share the dynamics models and the kinematics models.

The general mechanization theory [1-6, 11] describes how to acquire the position in either rectangular coordinates or geodetic coordinates, i.e., longitude, latitude and altitude.

There are at least two forms of algorithm about longitude, latitude and altitude, one is derivative equations, the other is conversion formulas from rectangular coordinates to geodetic ones in ECEF frame.

Both of the two forms of algorithm about longitude have singularities in the polar of the earth.

2. Theory About Longitude and Latitude

2.1. The Derivative Equations and Their Limits

The derivative equations describe the changing of longitude λ and latitude L as

$$\begin{cases} \dot{L} = v_N/(R_e + h) \\ \dot{\lambda} = v_E/[(R_e + h)cosL] \end{cases} \tag{1}$$

where v_N represents the northern projection of the velocity of the particle, v_E represents the eastern projection of the velocity of the particle, R_e represents the local radius of the Earth, h represents the altitude of the particle.

Obviously, the equation about longitude has singularities in polar of the Earth, where the latitude is $L = \pm 90°$ and the changing rate of longitude cannot be calculated for v_E is divided by zero.

2.2. Conversion Formulas from Rectangular Coordinates in the ECEF Frame

If the position of a particle in rectangular coordinates is acquired, a conversion formula can be used to calculate its longitude, latitude and altitude as

$$\begin{cases} h = \sqrt{x^2 + y^2 + z^2} - R_e \\ \tan\lambda = \dfrac{y}{x} \\ \sin L = \dfrac{z}{R_e + h} \end{cases} \tag{2}$$

where $x, y,$ and z represent the rectangular coordinates of a point in ECEF frame.

In polar, $x = 0, y = 0$, λ cannot be calculated either.

2.3. List of Algorithm About Transpolar Flying or Navigating

Some algorithms have been created and applied in transpolar flying or navigating [7-10]. Three popular algorithms are:

(1) Algorithm of transvers coordinate frame [14, 15];

(2) Algorithm of grid mechanization [12, 13];

(3) Algorithm of normal vector inertial navigation basing on earth-fixed frame [16].

An algorithm, which might be a new one, called the quaternion form of longitude, latitude and heading kinematics equations will be introduced in the paper.

3. Quaternion Form of Longitude Latitude and Heading Kinematics Equations

3.1. Review of the Concept of Great Circle

In the theories of navigation or flight simulation, great circle is one of the basic concepts.

A great circle can be defined as the circle within the surface of a sphere with the center same as the center of the sphere.

Obviously, the radius of any great circle is the greatest in all circles within the surface of a sphere.

In the Earth of absolute sphere model, all meridians are great circles, and the equator is the only great circle in all latitude circle. Besides, there are numberless leaning great circles neither a meridian nor the equator.

The position of a ship is always in the surface of the Earth. So, the concept of great circle is adequate to model the movement of a ship. But to an aircraft, its position might be out of any great circle. So, the concept of great circle needs to be expanded. An expanded concept based on the great circle is the flat surface of a great circle. That is a flat surface including the center of the Earth.

3.2. Declaration and Definition of an Instantaneous Great Circle

The instantaneous great circle declared in the paper might be a new concept.

Instantaneous great circles are a sub-aggregate of the aggregate of great circle. The distinguishing feature of an instantaneous great circle to a general great circle is that an instantaneous great circle is rotatable.

Observing a moving particle referenced to the Earth, we can determine a great circle by the tangent of its track and the center of the Earth except some special cases. The special cases include two. One is that the particle is static, the other is its tangent crosses the center of the Earth.

Firstly, analyze the general case. When a particle is moving continuously, the great circle decided by former rule always exists. So, the great circle is declared as an instantaneous great circle or an embroiled great circle.

Then, treat the two special cases. In the cases, instantaneous great circles are not non-existent, it is just un-definite. Or there are countless instantaneous great circle to the cases.

To these cases, it might as well to choose one of the countless instantaneous great circles. For example, the longitudinal axis of a ship's body frame can be chosen to determine the instantaneous great circle.

With continuous movement of a particle, its instantaneous great circle is moving too. Maybe we can describe that as a moving particle is driving its instantaneous great circle.

Obviously, the movement of an instantaneous great circle is pure rotation without translation in ECEF frame.

3.3. Definition of the Body Frame of an Instantaneous Great Circle

In order to model an instantaneous great circle, we'd better to define a 3-D frame for it. One of the definitions will be described as follows.

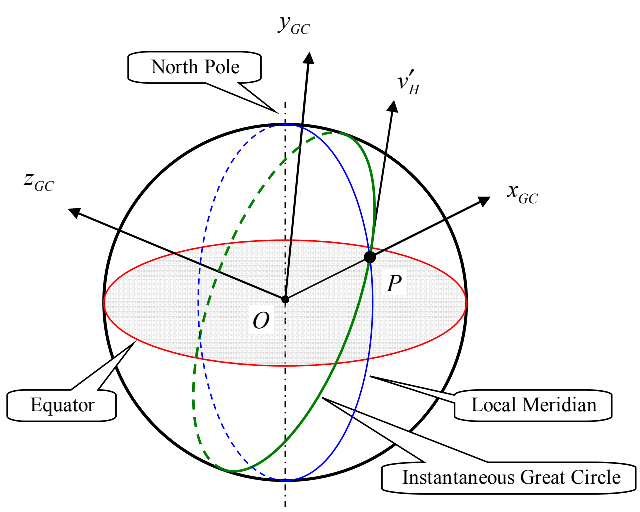

Figure 1. The Definition of the Body Frame of an Instantaneous Great Circle.

The origin of the frame is the center of the Earth. A line from the origin to the point of a particle is defined as the axis of x_{GC}. The line which is from the origin, perpendicular to x_{GC} and in the plane of the instantaneous great circle is defined as y_{GC}. The third axis z_{GC} is defined to a normal line of the plane of the instantaneous great circle, which is from the origin, and according to the rule of right hand.

The described definition of the body frame of an instantaneous great circle is shown in Figure 1.

3.4. Definition of the Euler Angles of an Instantaneous Great Circle

After defined a body frame for an instantaneous great circle, we can define three Euler angles for it to describe its attitude referenced to ECEF frame.

In possibility, there are several choices to define three Euler angles for an instantaneous great circle. After comparison, a choice called the directing definition was chosen. It is:

The longitude is defined as the heading of the instantaneous great circle, $\psi_{GC} = \lambda$;

The latitude is defined as the pitch angle, $\theta_{GC} = L$;

The heading of the horizontal velocity of a vehicle can be defined as the bank angle of the instantaneous great circle, $\phi_{GC} = \psi$.

The definition of Euler angles is shown in Figure 2.

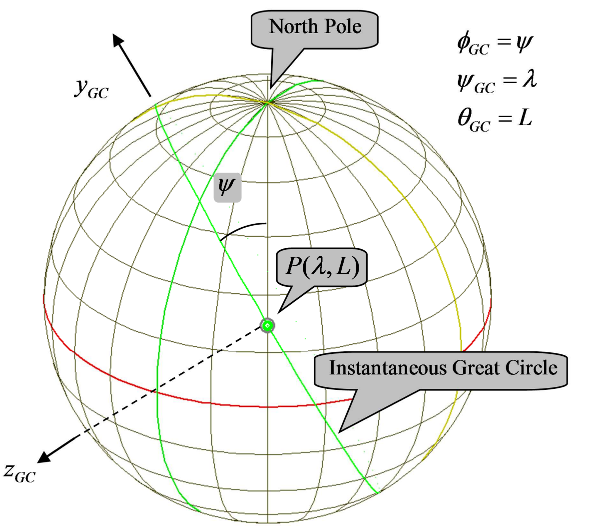

Figure 2. *Definition of the Euler Angles of an Instantaneous Great Circle.*

3.5. Analysis of the Angular Velocity of an Instantaneous Great Circle

Except for the center of the Earth, the vector of the velocity of a particle P always can be decomposed into two parts. One is the horizontal velocity v_H, the other is the vertical velocity v_V.

Meanwhile, there is acceleration vector of P. Decomposed the acceleration vector into 3-D orthogonal parts. One is parallel to the horizontal velocity, named as $a_{H\tau}$. The second is in the horizontal level but vertical to v_H, named as a_{Hs}, meaning lateral acceleration. The third is parallel to v_V, named as a_V.

The decomposition of the velocity and acceleration of P is shown in Figure 3.

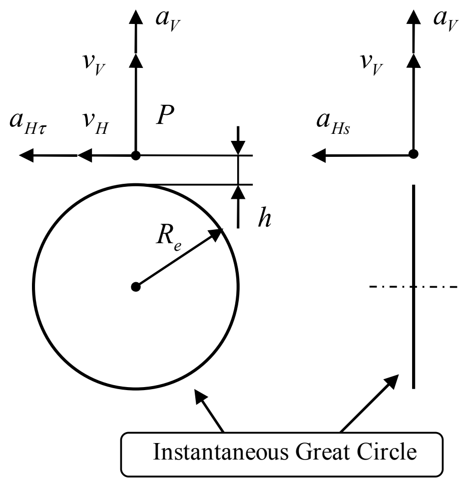

Figure 3. *Projections of the Vector of Velocity and Acceleration of P.*

Then let us try to analyze the rotation of the instantaneous great circle driven by its moving particle.

Vertical velocity v_V can only change the altitude of a particle. But if thinking of the self-rotation of the Earth, v_V can create Carioles acceleration except in polar. The direction of the Carioles acceleration is in horizontal plane. So, the possible effect of it will reflect in the horizontal acceleration;

Horizontal velocity v_H can directly drive its instantaneous great circle rotating around the axis z_{GC}. So, we get $\omega_{zGC} = v_H/(R_e + h)$;

Vertical acceleration a_V can change vertical velocity v_V without directly driving the instantaneous great circle;

Horizontal tangential acceleration a_{Ht} can only change horizontal velocity v_H without directly driving the instantaneous great circle;

Horizontal lateral acceleration a_{Hs} can change the direction or heading of horizontal velocity v_H, or make it yawing. The yawing equation is $v_H\dot{\psi}_s = a_{Hs}$.

As we all know, the angular rate $\dot{\psi}_s$ is different from $\dot{\psi}$. The latter one is the yaw rate of a rigid body or a coordinate fame, which can be measured by gyroscope. And the former one is the yaw rate of the horizontal velocity and the steady value of the latter. It can be calculated that based on measurement. The case is like relation of path angle and pitch angle.

Corresponding to $\dot{\psi}_s$, there is a curvature radius ρ of particle P when it is turning.

Their relationship is shown in Figure 4.

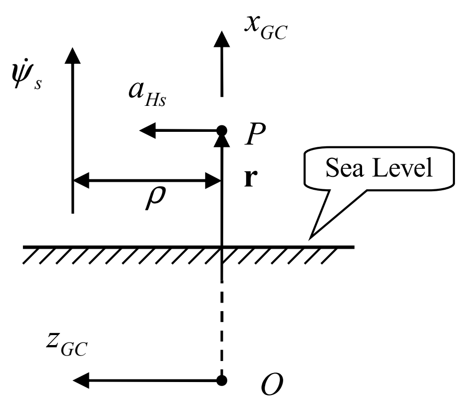

Figure 4. *Yaw of a Moving Particle.*

According to the theory of kinematics, there is an equation of vector form as

$$\dot{\psi}_s \times \rho = v_H \tag{3}$$

When the direction of v_H is invariant, $\dot{\psi}_s$ is zero, ρ is infinite. When a ship or an aircraft is turning, its curvature radius ρ will be in a range from hundreds of meters to thousands of meters promptly. Because the curvature radius ρ is not too big, we regard vector $\dot{\psi}_s$ is parallel to axis x_{GC}. So, the projection of $\dot{\psi}_s$ is completely on axis x_{GC}. That is

$$\omega_{xGC} = \dot{\psi} \tag{4}$$

Finally, we obtain the angular velocity of an instantaneous great circle as

$$\begin{bmatrix}\omega_{xGC}\\ \omega_{yGC}\\ \omega_{zGC}\end{bmatrix} = \begin{bmatrix}\dot{\psi}_s\\ 0\\ v_H/(R_e+h)\end{bmatrix} \tag{5}$$

Equation (5) can be applied for many kinds of vehicle but helicopters. Because there might be a case of moving laterally after a period of being static. In this case, before we define its instantaneous great circle by the vector of velocity, it is almost impossible for us to define an instantaneous great circle when it is static which can be continuous to the latter one.

To solve this case, let us change the definition of an instantaneous great circle a little. Referring to a rigid body has the property of inertia, thus it always rotates continuously under limited torture. So, it might as well to define the instantaneous great circle corresponding to the body frame of a vehicle. But considering of the longitudinal axis is not always horizontal, we can define a temporary frame based on a body frame. The temporary frame has the same heading as its body frame but has zero pitch and zero bank. We can define the instantaneous great circle by the temporary frame with the same heading of its body frame.

According to the definition, we get another angular velocity of and instantaneous great circle for a rigid body as

$$\begin{bmatrix}\omega_{xGC}\\ \omega_{yGC}\\ \omega_{zGC}\end{bmatrix} = \begin{bmatrix}\dot{\psi}\\ v_R/(R_e+h)\\ v_F/(R_e+h)\end{bmatrix} \tag{6}$$

Where v_R represents the right velocity, v_F represents the forwarding velocity.

It has been proved that (6) works well.

3.6. Rotation Equations of an Instantaneous Great Circle

After finishing the work of above, we can get rotation equations of an instantaneous great circle.

Directly, there are three rotation equations in a form of Euler angle. But the form includes two singularities when pitch angle is $\pm 90°$.

To solve the problem, we can use the theory of quaternion. Rotation equations in a form of quaternion work well without any singularity.

To a particle, as it has not three definite Euler angles, it is difficult to use quaternion theory to model it.

After we define an instantaneous great circle and three Euler angles to a particle, we can create a set of rotation equations in quaternion form to a moving particle or a rigid body. That is

$$\begin{bmatrix}\dot{q}_0\\ \dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\omega_{zGC}q_1 - \omega_{xGC}q_2 - \omega_{yGC}q_3\\ \omega_{zGC}q_0 + \omega_{yGC}q_2 - \omega_{xGC}q_3\\ \omega_{xGC}q_0 - \omega_{yGC}q_1 + \omega_{zGC}q_3\\ \omega_{yGC}q_0 + \omega_{xGC}q_1 - \omega_{zGC}q_2\end{bmatrix} \tag{7}$$

In (7), the vector $\{q_0, q_1, q_2, q_3\}$ represents the quaternion of a rigid body or a 3-D frame. It can describe the attitude of it. The quaternion is a substitute of Euler angles. The changing rate of a quaternion $\{\dot{q}_0, \dot{q}_1, \dot{q}_2, \dot{q}_3\}$ can describe the rotating of a rigid body or a 3-D frame. Obviously, there is not any singularity point in (7) all over the Earth.

Equations (7) is an algorithm about longitude, latitude and heading.

3.7. Corresponding Algorithm

The quaternion is not as direct as Euler angles indeed. The problem has been solved by the theory. There are two group of corresponding formulas in quaternion theory. The formulas for calculating quaternion form Euler angles are,

$$\begin{cases} q_0 = \cos\left(\frac{\lambda}{2}\right)\cos\left(\frac{L}{2}\right)\cos\left(\frac{\psi}{2}\right) - \sin\left(\frac{\lambda}{2}\right)\sin(\frac{L}{2})\sin(\frac{\psi}{2})\\ q_1 = \cos\left(\frac{\lambda}{2}\right)\cos\left(\frac{L}{2}\right)\sin\left(\frac{\psi}{2}\right) + \sin\left(\frac{\lambda}{2}\right)\sin(\frac{L}{2})\cos(\frac{\psi}{2})\\ q_2 = \cos\left(\frac{\lambda}{2}\right)\sin\left(\frac{L}{2}\right)\sin\left(\frac{\psi}{2}\right) + \sin\left(\frac{\lambda}{2}\right)\cos(\frac{L}{2})\cos(\frac{\psi}{2})\\ q_3 = \cos\left(\frac{\lambda}{2}\right)\sin\left(\frac{L}{2}\right)\cos\left(\frac{\psi}{2}\right) - \sin\left(\frac{\lambda}{2}\right)\cos(\frac{L}{2})\sin(\frac{\psi}{2}) \end{cases} \tag{8}$$

The formulas for calculating Euler angles form quaternion are:

$$\begin{cases} \psi = \arctan[-\frac{2(q_2q_3 - q_0q_1)}{q_0^2 - q_1^2 + q_2^2 - q_3^2}]\\ \lambda = \arctan[-\frac{2(q_1q_3 - q_0q_2)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}]\\ L = \arctan[2(q_1q_2 + q_0q_3)] \end{cases} \tag{9}$$

3.8. The Additional Acquirement of Heading Kinematics

By using the quaternion form of the equation, we have an additional acquirement of heading kinematics.

If it is unnecessary to calculate the longitude and latitude, we can regard the ground as an infinite plane. The heading of a vehicle or a vector projected in the plane will not be changed by the position changing. The heading equation is

$$\psi = \psi_0 + \int \dot{\psi} dt \tag{10}$$

Else if we need to calculate the longitude and latitude in a sphere frame, we have to face the question of a changing heading with the changing latitude and longitude.

Here are two examples. One is that when an aircraft is going to cross the north pole by flying straightly, its heading will not be constant, it will change rapidly. The other is that when an aircraft is circling around the north pole, its track projected in the ground will be a latitude circle. In the process, its heading is constant of east or west. But its yawing rate is not zero. So, it is somewhat difficult to model the heading movement.

After we create (7), the question of heading algorithm is solved naturally.

4. Some Results

Some study of mathematics simulation has been designed and done to verify the algorithm introduced above.

The simulating conditions are:

An aircraft flying toward north, western north, or eastern north from a point near north pole;

The aircraft will fly under the control of auto pilot with the mode of bank hold instead of heading hold or horizontal navigation;

One instruction degree of bank is 0. In the case, the aircraft will try to fly more than 40,000 km. To observe whether the simulating aircraft can complete a round-the-Earth by a great circle;

Another instruction degree of bank is about $\pm 20°$. To observe whether the simulating aircraft can fly around the north pole with a circle track.

Some results are shown as Figure 5 and Figure 6.

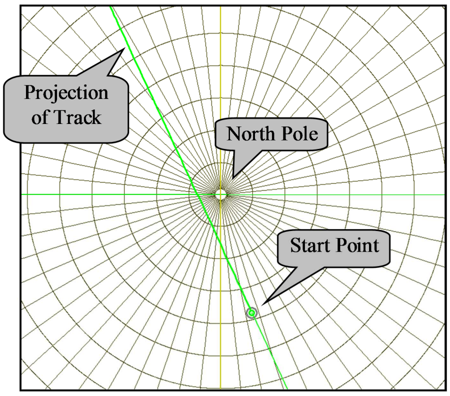

Figure 5. *Part of Horizontal Track of Round-the-Earth Flight.*

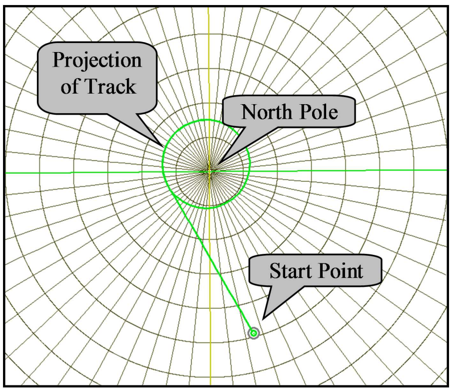

Figure 6. *Horizontal Track of Turning Around the North Pole by a Constant Bank Degree.*

The Figure 5 shows that the simulating aircraft can finish a perfect flight of round-the-Earth with the algorithm.

The Figure 6 show that the simulating aircraft can fly around the north pole in circle by a constant bank with the algorithm.

5. Conclusion

Theory of quaternion has been created for many years. It is succeeded in modeling rotation without any singularity which exist in Euler angles form.

To a moving particle, it is difficult to model its movement in a sphere frame by quaternion theory as there are not definite Euler angles to it.

If we define an instantaneous or embroil great circle for a moving particle or a rigid body, we can define three definite Euler angles for the great circle. Then we can create equations about longitude, latitude and heading in quaternion form without any singularity.

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