

Sharp Coefficient Related Results for Nephroid Shaped Domain

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To cite this article:

Dolly Jain. (2024). Sharp Coefficient Related Results for Nephroid Shaped Domain. *Pure and Applied Mathematics Journal*, 13(4), 51-58. <https://doi.org/10.11648/j.pamj.20241304.11>

Received: 24 June 2024; **Accepted:** 25 July 2024; **Published:** 7 August 2024

Abstract: In this study, the focus is on two main objectives related to starlike functions associated with a nephroid-shaped domain. Firstly, the aim is to determine sharp bounds for the coefficients of these functions up to the fifth order. These bounds are crucial as they provide a detailed understanding of the behavior of the coefficients, which is important for further analysis and various applications of these functions. The sharp determination of these coefficients can aid in refining mathematical models and theoretical frameworks involving starlike functions. Secondly, the sharp bound for the third order Hankel determinant for functions in this class is also derived. The Hankel determinant is a significant tool in complex analysis, as it provides insights into the growth, distortion, and other important properties of functions. By deriving these sharp bounds, this study improves upon the existing results in the literature, thereby contributing to a more sharp characterization of starlike functions associated with nephroid-shaped domains. This advancement has the potential to lead to enhanced applications, such as in geometric function theory and fluid dynamics, and offers a deeper understanding of these mathematical functions. By addressing these objectives, the study not only fills gaps in the current research but also opens new avenues for future exploration in the field of complex analysis.

Keywords: Coefficient Problems, Nephroid-shaped Domain, Starlike Functions, Hankel Determinants

1. Introduction

Let \mathcal{A} represent the class of normalized analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, given by the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

and let \mathcal{S} be a subclass of \mathcal{A} consisting of univalent (i.e., one-to-one) functions. Additionally, let \mathcal{P} denote the class of analytic functions on \mathbb{D} that have a positive real part, represented as $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Consider two analytic functions h and g . We say h is subordinate to g ($h \prec g$) provided there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| \leq |z|$ such that $h(z) = g(w(z))$. From the landmark Bieberbach conjecture of 1916 to contemporary studies (see [1]), a significant amount of literature is devoted to coefficient problems. In 1992, Ma and Minda [2] introduced

the following class which unified various subclasses of \mathcal{S}^* , defined by

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \quad (2)$$

where ϕ is an analytic univalent function satisfying the conditions $\operatorname{Re} \phi(z) > 0$, $\phi(\mathbb{D})$ symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$. Recently, many Ma-Minda classes are introduced and studied by several authors by appropriately choosing $\phi(z)$ such as e^z , $1 + 4z/3 + 2z^2/3$, $z + \sqrt{1+z^2}$ etc. in (2), see [3–5].

The concept of Hankel determinants, introduced in 1966 (see [6]), continues to be a topic of significant interest for researchers today. The definition of the q th Hankel determinant $H_{q,n}(f)$ of analytic functions $f \in \mathcal{A}$ while assuming $a_1 := 1$, is as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad n, q \in \mathbb{N}. \quad (3)$$

The expression for the third-order Hankel determinant for the specific values $q = 3$ and $n = 1$ is denoted by $H_{3,1}$, and is given by

$$H_{3,1}(f) := a_3(2a_2a_4 - a_3^2 + a_5) - a_4^2 - a_2^2a_5. \quad (4)$$

It is extremely difficult to derive a sharp bound for Hankel determinants, which is why numerous researchers have tried to do so for different subclasses of starlike functions, see [7–9] and the references cited therein for more details.

1.1. About the Class \mathcal{S}_{Ne}^*

In 2020, Wani and Swaminathan [10], introduced the Ma-Minda subclass of starlike functions \mathcal{S}_{Ne}^* by choosing $\phi(z) = 1 + z - z^3/3$, associated with a nephroid-shaped domain, defined by

$$\mathcal{S}_{Ne}^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3} \right\}.$$

They estimated various sharp radius related problems and also derived the radius of starlikeness for this domain. This class can be further studied to know more about the behavior of the coefficients of functions belonging to this class. Thus, in 2022, Kumar and Çetinkaya [11] obtained a possible bound of $H_{3,1}(f)$ as 0.925696 for functions f belonging to the class \mathcal{S}_{Ne}^* . Recently, Verma and Kumar [12] improved this bound to 0.395062. Therefore, taking inspiration from the above mentioned work, the sharp bound of the initial coefficients and third-order Hankel determinant for functions belonging to the class \mathcal{S}_{Ne}^* are established.

1.2. Preliminaries

In this part of the section, the sharp bounds of the initial coefficients a_i for $(i = 2, 3, 4, 5)$ are determined, followed by establishing the sharp bound of the third-order Hankel determinant for functions in \mathcal{S}_{Ne}^* .

Suppose $f \in \mathcal{S}_{Ne}^*$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = 1 + w(z) - \frac{w^3(z)}{3}. \quad (5)$$

Suppose that $p(z) = 1 + p_1z + p_2z^2 + \cdots \in \mathcal{P}$ and consider

$w(z) = (p(z) - 1)/(p(z) + 1)$. Further, by substituting the expansions of $w(z)$, $p(z)$ and $f(z)$ in (5) and then comparing the coefficients, the expressions of a_k ($k = 2, 3, 4, 5$) in terms of p_i ($i = 1, 2, \dots, 4$) are obtained, given as

$$a_2 = \frac{p_1}{2}, \quad a_3 = \frac{p_2}{4}, \quad a_4 = \frac{1}{72} \left(-p_1^3 - 3p_1p_2 + 12p_3 \right) \quad (6)$$

and

$$a_5 = \frac{1}{576} \left(5p_1^4 - 12p_1^2p_2 - 18p_2^2 - 24p_1p_3 + 72p_4 \right). \quad (7)$$

The results stated below are necessary for proving the main result.

Lemma 1.1. [14] Let $p \in \mathcal{P}$ be of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$|p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4| \leq 2 \quad (8)$$

and

$$|p_3 - 2p_1p_2 + p_1^3| \leq 2. \quad (9)$$

Lemma 1.2. [2] Let $p \in \mathcal{P}$ be of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$|p_2 - \beta p_1^2| \leq \begin{cases} 2 - 4\beta, & \beta \leq 0; \\ 2, & 0 \leq \beta \leq 1; \\ 4\beta - 2, & \beta \geq 1 \end{cases}$$

when $\beta < 0$ or $\beta > 1$, the equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. If $0 < \beta < 1$, then the inequality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $\beta = 0$, the equality holds if and only if $p(z) = (1+\eta)(1+z)/(2(1-z)) + (1-\eta)(1-z)/(2(1+z))$ ($0 \leq \eta \leq 1$) or one of its rotations. If $\beta = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in case of $\beta = 0$. Though the above upper bound is sharp for $0 < \beta < 1$, still it can be improved as follows:

$$|p_2 - \beta p_1^2| + \beta |p_1|^2 \leq 2 \quad (0 < \beta \leq 1/2) \quad (10)$$

and

$$|p_2 - \beta p_1^2| + (1-\beta)|p_1|^2 \leq 2 \quad (1/2 < \beta \leq 1).$$

Also, recall that

$$\max_{0 \leq t \leq 4} (At^2 + Bt + C) = \begin{cases} C, & B \leq 0, A \leq \frac{-B}{4}; \\ 16A + 4B + C, & B \geq 0, A \geq \frac{-B}{8} \\ & \text{or } B \leq 0, A \geq \frac{-B}{4}; \\ \frac{4AC - B^2}{4A}, & B > 0, A \leq \frac{-B}{8}. \end{cases} \quad (11)$$

The formula for p_k ($k = 2, 3, 4$), as stated in Lemma 1.3 is crucial in establishing sharp bounds for Hankel determinants and forms the cornerstone of the main results.

Lemma 1.3. [13, 14] Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then for some γ, η and ρ such that $|\gamma| \leq 1$, $|\eta| \leq 1$ and $|\rho| \leq 1$, provides

$$2p_2 = p_1^2 + \gamma(4 - p_1^2), \quad (12)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta,$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \bar{\gamma}\eta^2 - (1 - |\eta|^2)\rho).$$

2. Main Results

The sharp bounds of the initial coefficient upto fifth order are derived in the next theorem, given by

Theorem 2.1. If $f \in \mathcal{S}_{Ne}^*$, then (i) $|a_2| \leq 1$, (ii) $|a_3| \leq 1/2$, (iii) $|a_4| \leq 1/3$ and (iv) $|a_5| \leq 49/176 \simeq 0.611742 \dots$. These bounds are sharp.

Proof (i), (ii) Since $|p_n| \leq 2$ for $n \geq 1$, therefore, from (6), $|a_2| \leq 1$ and $|a_3| \leq 1/2$.

(iii) For a_4 , (5) is re-written as:

$$zf'(z) = \left(1 + w(z) - \frac{w^3(z)}{3}\right)f(z). \quad (13)$$

On substituting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $w(z) = \sum_{k=1}^{\infty} w_k z^k$ in (13) and comparing the coefficients of z^4 , gives

$$3a_4 = \left(w_3 + \frac{3}{2}w_1w_2 + \frac{w_1^3}{6}\right).$$

Now using [15], implies that $|3a_4| \leq 1$ and hence the desired bound is derived.

(iv) From (7), the expression of a_5 is as follows

$$\begin{aligned} a_5 &= \frac{1}{8} \left(\frac{5}{72}p_1^4 - \frac{1}{6}p_1^2p_2 - \frac{p_2^2}{4} - \frac{1}{3}p_1p_3 + p_4 \right) \\ &= \frac{1}{8} \left(-\frac{1}{4}L + \frac{1}{6}p_1M - \frac{7}{12}p_1^2N + \frac{3}{4}p_4 \right), \end{aligned}$$

which further gives

$$|a_5| \leq \frac{1}{8} \left(\frac{1}{4}|L| + \frac{1}{6}|p_1||M| + \frac{7}{12}|p_1|^2|N| + \frac{3}{4}|p_4| \right),$$

where $L = p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4$, $M = p_3 - 2p_1p_2 + p_1^3$ and $N = p_2 - (11/42)p_1^2$. Moreover, using the bounds of $|L| \leq 2$ from (8), $|M| \leq 2$ from (9) and $|N| \leq 2$ from (10), respectively, implies that

$$|a_5| \leq \frac{1}{8} \left(\frac{8}{3} + \frac{7}{6}|p_1|^2 - \frac{11}{72}|p_1|^4 \right).$$

Now, it is to be noted that $|7|p_1|^2/4 - 469|p_1|^4/1152| \leq 49/22$ using (11) by taking $A = -11/72$, $B = 7/6$ and $C = 0$, which leads to the desired estimate for $|a_5|$. The sharpness of the result can be witnessed when $p_1 = p_4 = 2$, $p_2 = -1.33 + 0.142134i$ and $p_3 = -2$. The function

$$f_n(z) = z \exp \left(\int_0^z \frac{t^k - t^{3k}}{t} dt \right)$$

acts as the extremal function for the initial coefficients a_2 and a_3 for $k = 1$ and a_4 for $k = 3$.

The following theorem presents the sharp bound for the third-order Hankel determinant for functions belonging to the class \mathcal{S}_{Ne}^* .

Theorem 2.2. Let $f \in \mathcal{S}_{Ne}^*$, then

$$|H_{3,1}(f)| \leq \frac{1}{36}. \quad (14)$$

This result is sharp.

Proof Since the class \mathcal{P} is invariant under rotation, implies that $p_1 =: p$ belongs to the interval $[0, 2]$. Substitute the values of a_i ($i = 2, 3, 4, 5$) in (4) from (6) and (7), implies that

$$H_{3,1}(f) = \frac{1}{20736} \left(-49p^6 + 57p^4p_2 - 198p^2p_2^2 - 486p_2^3 + 312p^3p_3 + 936pp_2p_3 - 576p_3^2 - 648p^2p_4 + 648p_2p_4 \right).$$

After simplifying the calculations using Lemma 1.3, consider

$$H_{3,1}(f) = \frac{1}{82944} \left(h_1(p, \gamma) + h_2(p, \gamma)\eta + h_3(p, \gamma)\eta^2 + h_4(p, \gamma, \eta)\rho \right),$$

for $\gamma, \eta, \rho \in \mathbb{D}$. Here

$$\begin{aligned} h_1(p, \gamma) &:= -49p^6 - 81\gamma^2p^2(4-p^2)^2 - 324\gamma^3(4-p^2)^2 - 135\gamma^3p^2(4-p^2)^2 + 18\gamma^4p^2(4-p^2)^2 \\ &\quad + 117\gamma p^4(4-p^2) - 6p^4\gamma^2(4-p^2) - 162p^4\gamma^3(4-p^2) - 648\gamma^2p^2(4-p^2), \\ h_2(p, \gamma) &:= 24(1-|\gamma|^2)(4-p^2)(14p^3 + 27\gamma p^3 + 18p\gamma(4-p^2) - 3p\gamma^2(4-p^2)), \\ h_3(p, \gamma) &:= 72(1-|\gamma|^2)(4-p^2)(-8(4-p^2) - |\gamma|^2(4-p^2) + 9p^2\bar{\gamma}), \\ h_4(p, \gamma, \eta) &:= 648(1-|\gamma|^2)(4-p^2)(1-|\eta|^2)((4-p^2)\gamma - p^2). \end{aligned}$$

By choosing $r = |\gamma|$, $s = |\eta|$ and utilizing the fact that $|\rho| \leq 1$, the above expression reduces to the following:

$$|H_{3,1}(f)| \leq \frac{1}{82944} \left(|h_1(p, \gamma)| + |h_2(p, \gamma)|s + |h_3(p, \gamma)|s^2 + |h_4(p, \gamma, \eta)| \right) \leq A(p, r, s),$$

where

$$A(p, r, s) = \frac{1}{82944} \left(a_1(p, r) + a_2(p, r)s + a_3(p, r)s^2 + a_4(p, r)(1-s^2) \right), \quad (15)$$

with

$$\begin{aligned} a_1(p, r) &:= 49p^6 + 81r^2p^2(4-p^2)^2 + 324r^3(4-p^2)^2 + 135r^3p^2(4-p^2)^2 + 18r^4p^2(4-p^2)^2 \\ &\quad + 117rp^4(4-p^2) + 6p^4r^2(4-p^2) + 162p^4r^3(4-p^2) + 648r^2p^2(4-p^2), \\ a_2(p, r) &:= 24(1-r^2)(4-p^2)(14p^3 + 27rp^3 + 18pr(4-p^2) + 3pr^2(4-p^2)), \\ a_3(p, r) &:= 72(1-r^2)(4-p^2)(8(4-p^2) + r^2(4-p^2) + 9p^2r), \\ a_4(p, r) &:= 648(1-r^2)(4-p^2)(r(4-p^2) + p^2). \end{aligned}$$

In the closed cuboid $Q : [0, 2] \times [0, 1] \times [0, 1]$, the aim is to maximise $A(p, r, s)$, by identifying the maximum values within the interior of the six faces, along the twelve edges, and inside the interior of Q .

1. Consider the interior of each of the six faces of the cuboid Q .

On $p = 0$

$$b_1(r, s) := \frac{1}{288} \left(18r^3 + 4(1-r^2)(8+r^2)s^2 + 36r(1-r^2)(1-s^2) \right), \quad r, s \in (0, 1). \quad (16)$$

Since

$$\frac{\partial b_1}{\partial s} = \frac{(1-r)^2(r+1)(r-8)s}{36} \neq 0, \quad r, s \in (0, 1),$$

suggests that b_1 does not contain any critical points in $(0, 1) \times (0, 1)$.

On $p = 2$

$$A(2, r, s) := \frac{49}{1296} = 0.0378086, \quad r, s \in (0, 1). \quad (17)$$

On $r = 0$

$$b_2(p, s) := \frac{1}{82944} \left(49p^6 + (4-p^2)(336p^3s + 576s^2(4-p^2) + 648p^2(1-s^2)) \right) \quad (18)$$

where $p \in (0, 2)$ and $s \in (0, 1)$. To find the points of maxima, solve $\partial b_2/\partial p = 0$ and $\partial b_2/\partial s = 0$. The partial differentiation $\partial b_2/\partial s = 0$, implies that

$$s = \frac{7p^3}{3(17p^2 - 32)} =: s_p. \quad (19)$$

To ensure that $r_p \in (0, 1)$ for the given range of r , it is necessary that $p_0 := p > \approx 1.54572$. After calculations, $\partial b_2/\partial p = 0$ gives

$$0 = 5184p - 2592p^3 + 294p^5 + 4032p^2r - 1680p^4r - 14400pr^2 + 4896p^3r^2 = 0. \quad (20)$$

After substituting (19) in (20), gives

$$0 = 5308416p - 8294400p^3 + 4318272p^5 - 861984p^7 + 44982p^9. \quad (21)$$

Based on numerical calculations, $p \approx 1.16654 \in (0, 2)$ is found out to be the solution of (21). Therefore, b_2 does not assume any critical point in $(0, 2) \times (0, 1)$.

On $r = 1$

$$b_3(p, r) := \frac{2592 + 1872p^2 - 528p^4 - p^6}{41472}, \quad p \in (0, 2). \quad (22)$$

While computing $\partial b_3/\partial p = 0$, $p_0 := p \approx 1.32811$ appears as the critical point. Through straightforward calculations, b_3 reaches its maximum value of approximately ≈ 0.102376 at the critical point.

On $r = 0$

$$b_4(p, r) := \frac{1}{82944} \left(5184r(2 - r^2) + 144p^2(18 - 36r + 9r^2 + 33r^3 + 2r^4) - 12p^4(54 - 93r + 52r^2 + 63r^3 + 12r^4) + p^6(49 - 117r + 75r^2 - 27r^3 + 18r^4) \right).$$

After further calculations such as,

$$\frac{\partial b_4}{\partial r} = \frac{(4 - p^2)}{27648} \left(432(2 - 3r^2) - 24p^2(9 - 9r - 36r^2 - 4r^3) + p^4(39 - 50r + 27r^2 - 24r^3) \right)$$

and

$$\frac{\partial b_4}{\partial p} = \frac{1}{13824} \left(48p(18 - 36r + 9r^2 + 33r^3 + 2r^4) - 8p^3(54 - 93r + 52r^2 + 63r^3 + 12r^4) + p^5(49 - 117r + 75r^2 - 27r^3 + 18r^4) \right),$$

observe that only real solutions (p, r) of the system of equations $\partial k_4/\partial r = 0$ and $\partial k_4/\partial p = 0$ are $(5.19778, 2.1876)$, $(-1.1126, -1.21914)$, $(-2, -0.126237)$, $(-5.19778, 2.1876)$, $(2, -0.126237)$, $(0, -0.816497)$, $(0, 0.816497)$, and $(1.1126, -1.21914)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

On $r = 1$

$$b_5(p, r) := \frac{1}{82944} \left(1152pr(6 + r - 6r^2 - r^3) + 576(16 - 14r^2 + 9r^3 - 2r^4) - 24p^5(14 + 9r - 17r^2 - 9r^3 + 3r^4) + 144p^2(-32 + 18r + 55r^2 - 21r^3 + 6r^4) + 96p^3(14 - 9r - 20r^2 + 9r^3 + 6r^4) + 12p^4(48 - 15r - 148r^2 + 45r^3 - 18r^4) + p^6(49 - 117r + 75r^2 - 27r^3 + 18r^4) \right).$$

Simple calculations leads to

$$\frac{\partial b_5}{\partial r} = \frac{(4-p^2)}{27648} \left(-48r(28-27r+8r^2) + 192p(3+r-9r^2-2r^3) + 24p^2(9+41r-18r^2+8r^3) \right. \\ \left. + 8p^3(9-34r-27r^2+12r^3) + p^4(39-50r+27r^2-24r^3) \right)$$

and

$$\frac{\partial b_5}{\partial p} = \frac{1}{13824} \left(192r(6+r-6r^2-r^3) - 20p^4(14+9r-17r^2-9r^3+3r^4) \right. \\ \left. + 48p(-32+18r+55r^2-21r^3+6r^4) + 48p^2(14-9r-20r^2+9r^3+6r^4) \right. \\ \left. - 8p^3(-48+15r+148r^2-45r^3+18r^4) + p^5(49-117r+75r^2-27r^3+18r^4) \right).$$

Note that the only real solutions (p, r) of the system of equations $\partial k_5/\partial r = 0$ and $\partial k_5/\partial p = 0$ are $(-1.45719, -20.4089)$, $(2.64352, 16.5905)$, $(2, 1.38132)$, $(1.2031, 0.707224)$, $(-1.67605, -0.63288)$, $(-0.360067, 0.884355)$, $(-2, 0.772396)$, $(0, 0)$, $(2, -1.34647)$ and $(-1.62288, 0.165777)$. Thus, no solution exists in $(0, 2) \times (0, 1)$, resulting in no critical points.

2. Next, analyze the maximum values achieved by $A(p, r, s)$ on the edges of the cuboid Q .

By (18), $A(p, 0, 0) = w_1(p) := (2592p^2 - 648p^4 + 49p^6)/82944$. It is observed that $w_1'(p) = 0$ only at $p = \alpha_0 := 0$ and $p = \alpha_1 := 1.75123 \in [0, 2]$ as the points of minima and maxima respectively. Thus,

$$A(p, 0, 0) \leq 0.0393988, \quad p \in [0, 2].$$

Now considering (18) at $r = 1$, gives $A(p, 0, 1) = w_2(p) := (96 - 24p^2 + 7p^3)^2/82944$. Note that $w_2'(p) < 0$ in $[0, 2]$ indicating that $p = 0$ is the point of maxima. Thus,

$$A(p, 0, 1) \leq \frac{1}{9} \approx 0.111111, \quad p \in [0, 2].$$

Simple calculations, (18) reveals that $A(0, 0, s)$ attains its maximum at $r = 1$. This implies that

$$A(0, 0, s) \leq \frac{1}{9}, \quad r \in [0, 1].$$

Since, (22) does not involve r , provides $A(p, 1, 1) = A(p, 1, 0) = w_3(p) := (2592 + 1872p^2 - 528p^4 - p^6)/41472$. Now, $w_3'(p) = 624p - 354p^3 - p^5 = 0$ when $p = \alpha_2 := 0$ and $p = \alpha_3 := 1.32811$ in the interval $[0, 2]$ with α_2 and α_3 as points of minima and maxima, respectively. Hence

$$A(p, 1, 1) = A(p, 1, 0) \leq 0.0948833, \quad p \in [0, 2].$$

After considering $p = 0$ in (22), implies that $A(0, 1, s) = 1/64$. Equation (17) does not involve any variable. Therefore, on the edges, the maxima of $M(p, r, s)$ for $r, s \in [0, 1]$ is

$$A(2, 1, s) = A(2, 0, s) = A(2, r, 0) = A(2, r, 1) = \frac{49}{1296}.$$

Using (16), $A(0, r, 1) = w_4(r) := (16 - 14r^2 + 9r^3 - 2r^4)/144$. After calculations, it is evident that $w_4(r)$ is a decreasing function in $[0, 1]$ and achieves its maximum when $r = 0$. Thus

$$A(0, r, 1) \leq \frac{1}{9}, \quad r \in [0, 1].$$

Again using (16), implies that $A(0, r, 0) = w_5(r) := r(2 - r^2)/16$. Upon further calculation, $w_5'(r) = 0$ gives $r = \alpha_4 := \sqrt{2/3}$. Additionally, $w_5(r)$ is an increases in $[0, \alpha_4)$ and decreases in $(\alpha_4, 1]$. Therefore, it reaches its maximum value at α_4 . Thus

$$A(0, r, 0) \leq 0.0680414, \quad r \in [0, 1].$$

3. At last, consider every internal point of Q . Assume that $(p, r, s) \in (0, 2) \times (0, 1) \times (0, 1)$. By calculating $\partial A/\partial r$, partially

differentiate (15) with respect to r to identify the points of maxima in the interior of Q , gives

$$\frac{\partial A}{\partial r} = \frac{(4-p^2)(1-r^2)}{3456} \left(12pr(6+r) + p^3(14+9r-3r^2) - 6p^2(17-18r+r^2)r + 24(8-9r+r^2)r \right).$$

Now $\partial A/\partial r = 0$ gives

$$r = r_0 := \frac{12pr(6+r) + p^3(14+9r-3r^2)}{6(p^2(r-17) - 4(r-8))(r-1)}.$$

The existence of critical points requires that r_0 belong to $(0, 1)$, which is only possible when

$$102p^2 + 216x + 3p^3x^2 + 6p^2x^2 > 192 + 108p^2x + 24x^2 + 14p^3 + 72px + 9p^3x + 12px^2. \quad (23)$$

Next, determine the solution that satisfies the inequality (23) for the existence of critical points using the hit and trial method. Assuming p tends to 0, note that there does not exist any $r \in (0, 1)$ satisfying (23). But, when p tends to 2, (23) holds for all $r < 13/54$. Also, observe that there does not exist any $p \in (0, 2)$ when $r \in (13/54, 1)$. Similarly, if r tends to 0, then for all $p > 1.54572$, (23) holds. After calculations, it is to be observed that there does not exist any $r \in (0, 1)$ when $p \in (0, 1.54572)$. Thus, the domain for the solution of the inequality is $(1.54572, 2) \times (0, 13/54)$. Further, note that $\frac{\partial A}{\partial r}|_{r=r_0} \neq 0$ in $(1.54572, 2) \times (0, 13/54)$. Hence, the function A has no critical point in $(0, 2) \times (0, 1) \times (0, 1)$.

Observing all the three cases, the inequality (14) holds. Define the function $f_0(z) \in \mathcal{S}_{N_e}^*$ as follows:

$$f_0(z) = z \exp \left(\int_0^z \frac{3t^3 - t^9}{3t} dt \right) = z + \frac{z^4}{3} + \frac{z^7}{18} + \cdots,$$

with $f_0(0) = 0$ and $f_0'(0) = 1$, serves the role of an extremal function for $|H_{3,1}(f)|$.

3. Conclusion

In this paper, the class of starlike functions associated with a nephroid-shaped domain is examined. This work led to the determination of sharp bounds for the initial coefficients up to the fifth order, providing a strong foundation for future studies and applications of this class. Furthermore, it led to achieve a significant improvement in the bound of the third-order Hankel determinant, reducing it from 0.925696 and 0.395062 to 0.1111. This new bound is the best possible for functions belonging to this class, marking a substantial advancement in the mathematical understanding and characterization of starlike functions associated with nephroid-shaped domains. These results pave the way for deeper investigations and potentially new applications of these functions in various fields.

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Conflicts of Interest

The author declare no conflicts of interest.

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